Waves and Superposition (Keating Chapter 21)

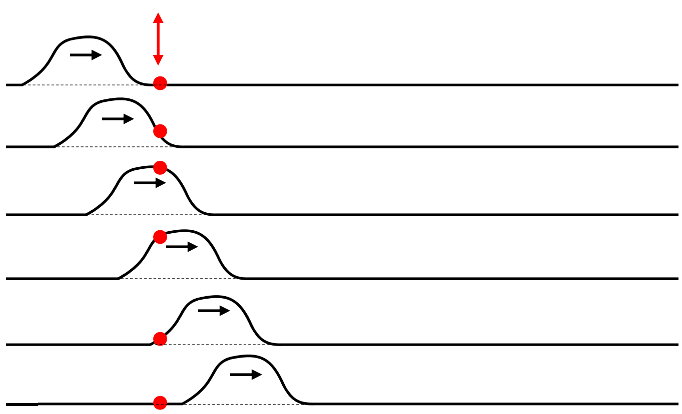
The ray model for light (i.e. light travels in straight lines) can be used to explain a lot of phenomena (like basic object and image formation and even aberrations) but it can't explain some very important phenomena like:
•interference
•diffraction
•polarization

To explain these phenomena we adopt a new model to describe how light travels. Light, electricity and magnetism are fundamentally linked. Excited electrically charged particles radiate energy which travels as waves. The energy that radiates is called electromagnetic energy. One component is called the electric vector and another is called the magnetic vector. Each electric field is associated with it own magnetic field which together makes up what is called **electromagnetic radiation**. The electric component of this energy is sufficient to explain the phenomena we'll describe in this section of the course so we will only concern ourselves with this.

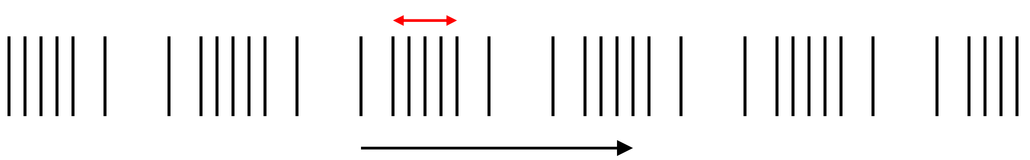
Radiating light behaves as a wave. That means that one can think of light like one thinks of ripples in a pond. It interacts with itself in the same way. Remember that this is only a model to explain how light behaves. It is analogous to the reason that we adopt a model of rays to describe how light forms images.

Light as a Transverse Wave.

To model light properly we have to treat it as a transverse wave. In a transverse wave, energy propagates forward but the oscillation of the wave is perpendicular to the direction of motion. Like a wave in a rope; no part of the rope moves forward or backward, only up and down but the wave (or energy) travels longitudinally. You can move energy without transporting the media. Similarly, light doesn't push that air that it travels through.



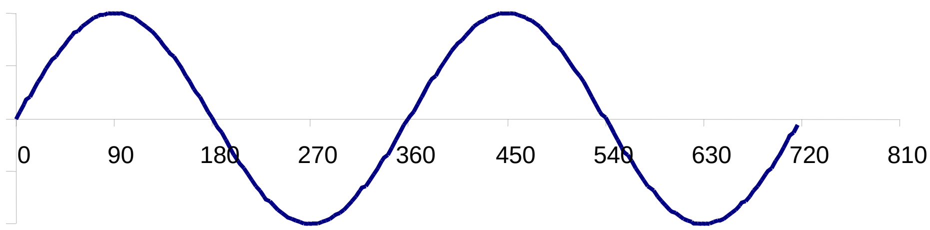
This is different from a pressure wave that sound produces which is a longitudinal wave



We need to model light as a **transverse** wave to be able to explain **polarization**. Like rope, the disturbance in light travels in one direction. That direction is called the direction of polarization of the light. Sound waves, being a pressure wave, cannot be polarized. We will discuss polarization in a later lecture.

Unlike the transverse wave in a rope, transverse light waves travel in all directions in three dimensions.

When a pulse, like in a rope is repeated continually, then a periodic wave is formed. Light is most often modeled as a periodic wave since it arises from continually vibrating (accelerating) electrically-charged particles. As such, light can be modeled as a continuous sine wave. The sine wave is also called a **harmonic wave** since the motion of any one point on the wave over time undergoes **simple harmonic motion.**



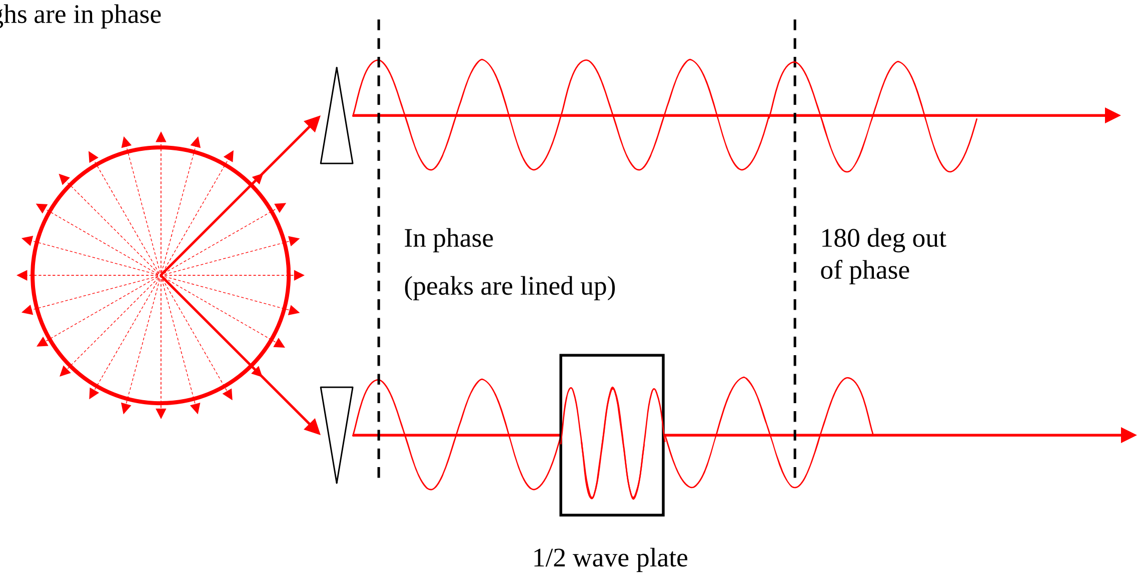
$$y = A\sin p$$

The above equation generally defines a light wave, where $\boldsymbol{A}$ is the amplitude or maximum displacement (related to intensity) and $\boldsymbol{p}$ is the phase. The phase describes what part of the cycle the transverse disturbance is in. When the phase is 0, y = 0, when the phase is 90, y = A etc… The distance between the peaks of two waves is often described as a phase shift. It is measured in degrees or radians

$$1\lambda = 360° = 2\pi \text{ radians}$$
$$\tfrac{1}{2}\lambda = 180° = \pi \text{ radians}$$
$$\tfrac{1}{4}\lambda = 90° = \tfrac{\pi}{2} \text{ radians}$$

When corresponding points on two different waves are moving in the same direction and are at the same point in their cycle, they are said to be **in phase**

All the points that join the crest of a wave of light emanating from a point source are in phase. Likewise the troughs are in phase



Some Important Definitions

Wavelength (λ): The distance between two adjacent crests on the wave

Period (T): The time it takes for a point on a moving wave to go through one complete up and down cycle

Frequency (*f*): The number of cycles per unit time is called the frequency.

$$f = \frac{1}{T}$$

The units for frequency in cycles per second is called the Hertz or Hz.

The distance a wave moves during a period T is one wavelength, therefore:

$$\lambda = \mathrm{V} \cdot \mathrm{T}$$

Then it follows from the above two equations that wavelength times the frequency equals the speed:

$$\lambda f = \mathrm{V}$$

The irradiance delivered by a light wave is proportional to the square of the amplitude of the wave:

$$\mathrm{I} \, \alpha \, \mathrm{A}^2$$

Frequency Invariance

Light can be defined by its wavelength, but since it is a propagating wave, it also has a frequency. I had already stated that the speed of light changes as it goes through different media. The frequency, on the other hand, remains constant. As a consequence, the wavelength of light must change as light propagates through different media in order to satisfy the equation $\lambda f = V$. When we classify colors in terms of wavelength, we use the wavelength in a vacuum (or effectively, in air). Remember that the velocity of light is equal to c/n. Combining that with the fact the frequency is constant, we can calculate just how the wavelength changes in different media.

$$c = 3 \times 10^8 \text{ m/s}$$

$$\mathrm{V} = \frac{c}{n}, \quad \text{recall that } \mathrm{f} \cdot \lambda_{\mathrm{m}} = \mathrm{V} = \frac{c}{n}$$

We will use the subscript *m* to designate the wavelength of light in some medium other than a vacuum (or air)

$$\lambda_m = \frac{c}{f \cdot n}, \text{ but the wavelength in a vacuum is } \frac{c}{f}, \quad \therefore$$

$$\lambda_{\mathrm{m}} = \frac{\lambda}{n}$$

$$\mathrm{f} = \frac{\mathrm{c}}{\lambda}, \quad \text{for red light (633 nm)}, \quad = \frac{3 \times 10^8}{633 \times 10^{-9}} = 4.74 \times 10^{14} \text{ Hz}$$

when red light is in the eye, the wavelength is

$$\lambda_{\mathrm{m}} = \frac{633}{1.33} = 474 \text{ nm}$$

Even though we speak of color in terms of its wavelength (ie blue is 450 nm), it is more appropriate to speak of color in terms of its frequency. Because although the wavelength of red light changes in the eye to 474, it still appears red. We are sensitive to the frequency, which is invariant, not the wavelength

Comments on the Nature of Light and the Wave Model of Light

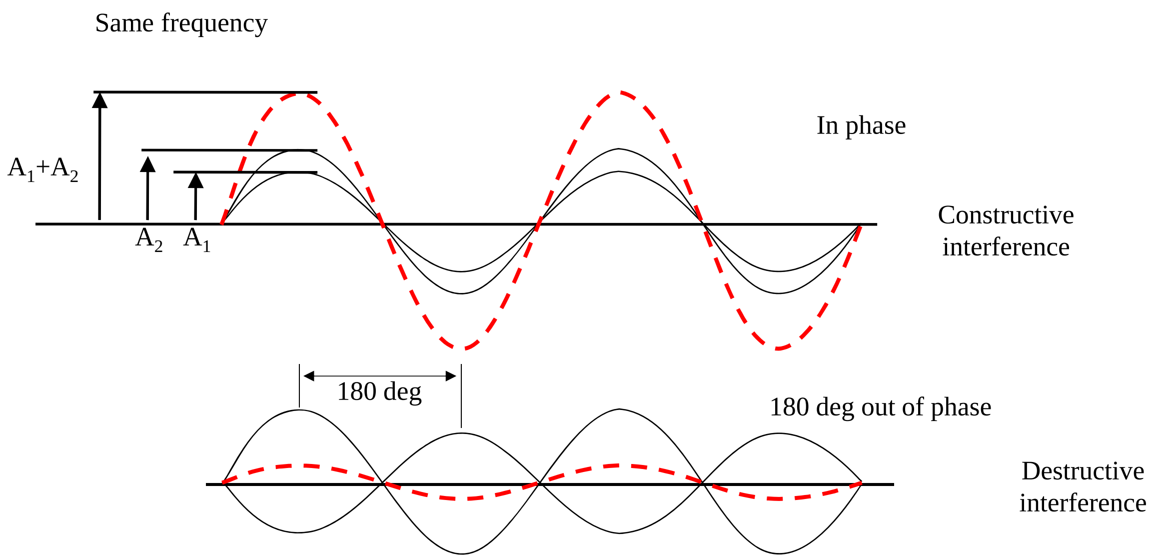
When I keep repeating that the description of light as a wave is a model, it is for a reason.

Light can't really be a wave since a real wave requires a medium through which to propagate, like air or water. But light travels in a vacuum. Huygens, in 1690, invented the concept of **ether** as a medium which fills all space to account for this, but space is indeed a vacuum so his theory fell apart.

So in some cases, such as to explain how light can travel through a vacuum, we must consider light as a particle. But when you think of light as a particle of energy, how is it possible that this light can undergo partial reflection at a glass surface? Or how can identical particles suffer different fates when following the same path? This is referred to as the **wave-particle** duality of light, which has been partially explained by quantum theory but it is still a very confusing concept, which has never been explained fully.

The Principle of Superposition

Waves add together by their amplitude:



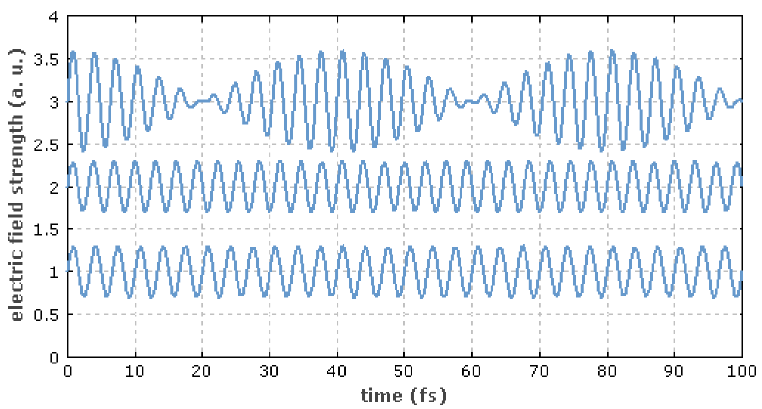
$$y_r = A_1 \sin p_1 + A_2 \sin p_2$$

when the phase is the same,

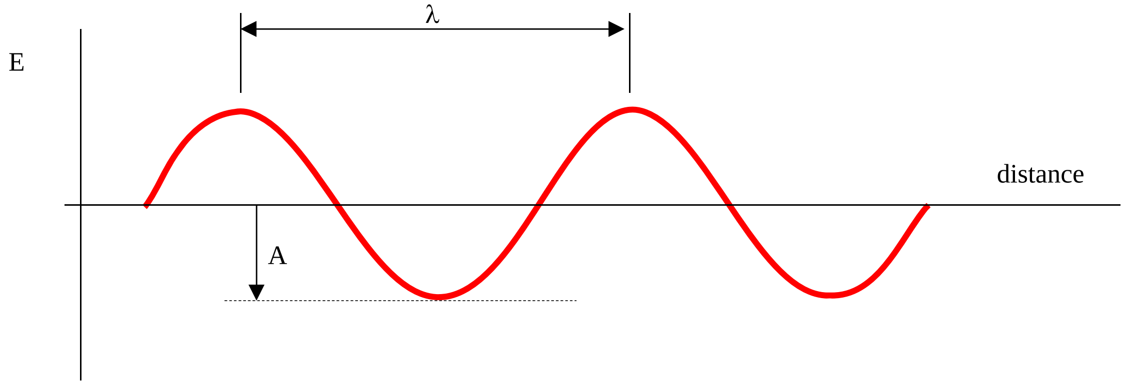
$$y_r = A_1 \sin p_1 + A_2 \sin p_1 = \left(A_1 + A_2\right)\sin p_1$$

when the phase is out by 180 deg,

$$y_r = A_1 \sin p_1 + A_2 \sin\left(p_1 + 180\right) = A_1 \sin p_1 - A_2 \sin p_1 = \left(A_1 - A_2\right)\sin p_1$$



When two waves have slightly different frequencies, then beat frequencies are generated as illustrated above . This is one of the methods that are used to generate extremely short-pulsed lasers.



The moving wave is represented by:

$$E = A \cdot \sin(kx - \omega t + \varepsilon)$$

Where:

$$k = \frac{2\pi}{\lambda} \equiv \text{propagation number} \qquad (1)$$

$$\omega = 2\pi f \equiv \text{angular frequency} \qquad (2)$$

$$\varepsilon \equiv \text{initial phase [in radians]} \qquad (3)$$

The phase term *p* that was used previously is a number that varies in space and in time and can have some fixed offset. The term *kx* describes how the wave changes over space, the term ωt describes how the wave changes in time. The term ε describes the offset, or phase shift.

$$\text{from (1) and (2),} \qquad \frac{\omega}{k} = f\lambda = \text{V}$$

Dissection of the Wave Equation:

If we freeze the moving wave in time (hold *t* constant) then we can find the amplitude at any point in the wave by varying x. Conversely, at any point in space, we can find how the amplitude changes in time by holding *x* constant and varying t. When ω is positive, the wave is traveling to the right, when ω is negative, the wave is traveling to the left.

The function is periodic ie: $\sin(\theta) = \sin(\theta \pm 2\pi m)$

$$m = 0, \pm 1, \pm 2 \ldots$$

So, if I give you the wave equation, you can determine the wavelength, the initial phase (at time and location 0) and the speed of the wave. Conversely, you should be able to develop the wave equation for any wavelength if I provide the initial phase and the direction of propagation.

The intensity of a wave is the square of the amplitude of the wave equation:

$$I \alpha A^2$$

Consider two waves:

$$E_1 = A_1 \cdot \sin(k_1 x - \omega_1 t + \varepsilon_1)$$
$$E_2 = A_2 \cdot \sin(k_2 x - \omega_2 t + \varepsilon_2)$$

The two waves are said to **mutually coherent** if the crest of the first wave is always a fixed distance from the crest of the second wave. This will only happen when $k_1 = k_2$, $\omega_1 = \omega_2$ and $\varepsilon_1 - \varepsilon_2$ is a constant.

When two waves are mutually coherent they can interfere in the ways we described earlier. To add the waves mathematically, we must first add the amplitudes, then square the result to find the intensity.

For two **mutually coherent** waves, we can simplify by writing:

$$E_1 = A_1 \cdot \sin(p_1)$$
$$E_2 = A_2 \cdot \sin(p_2)$$
$$E_1 + E_2 = A_1 \cdot \sin(p_1) + A_2 \cdot \sin(p_2)$$
$$= A_r \sin(p_r)$$

The resultant wave has a new amplitude and a new phase but has the same frequency and wavelength

This equation is obtained after some algebra (which you are not responsible for)

$$I_{coherent} = (E_1 + E_2)^2 = A_1^2 + A_2^2 + 2 \cdot A_1 A_2 \cos(p_1 - p_2)$$

$$\text{when } A_1 = A_2 \quad \Rightarrow \quad A_r^2 = 4A^2 \cos^2\left(\frac{p_1 - p_2}{2}\right)$$

For completeness, the phase can also be calculated using the following equation, but you will not be responsible to remember this on the exam.

$$\tan(p_r) = \frac{A_1 \sin p_1 + A_2 \sin p_2}{A_1 \cos p_1 + A_2 \cos p_2}$$

Example:

What is the intensity of two mutually coherent waves, one with amplitude 5 and another with amplitude 13 and a phase difference between the two of 90 degrees? 180 degrees?

$$I_{coherent} = (E_1 + E_2)^2 = A_1^2 + A_2^2 + 2 \cdot A_1 A_2 \cos(p_1 - p_2)$$

for 90 deg phase difference...

$$I_{coherent} = 5^2 + 13^2 + 2 \cdot 5 \cdot 13 \cdot \cos(90) = 25 + 169 + 0 = 194$$

for 180 deg phase difference...

$$I_{coherent} = 5^2 + 13^2 + 2 \cdot 5 \cdot 13 \cdot \cos(180) = 25 + 169 - 130 = 64$$

recall that when the waves are 180 deg out of phase, then the resultant amplitude is the difference between the two waves

$$\sqrt{64} = 8$$

When the phase difference between two waves varies in time and space, the waves are said to be **mutually incoherent**. In this case, the instant that a wave can interact is immediately followed by an instant in which the waves have no correlation. Coherent light sources generally originate from the same light source. For example, coherent interactions are observed with lasers when the light is split and interacts with itself. This is because both waves start and emerge from the same source, the wavelength bandwidth is very narrow. Even though the initial phase might get perturbed slightly, the same changes are experienced in both paths.

For **incoherent light** you simply add the independently calculated intensity of each of the light sources.

$$I_{incoherent} = E_1^{\ 2} + E_2^{\ 2} = A_1^{\ 2} + A_2^{\ 2}$$

The necessary condition for two sources of light to interfere is that they be **mutually coherent**. (i.e the crest of one wave is always a fixed distance from the crest of the other wave). In practice, these two sources must originate from the same light source.

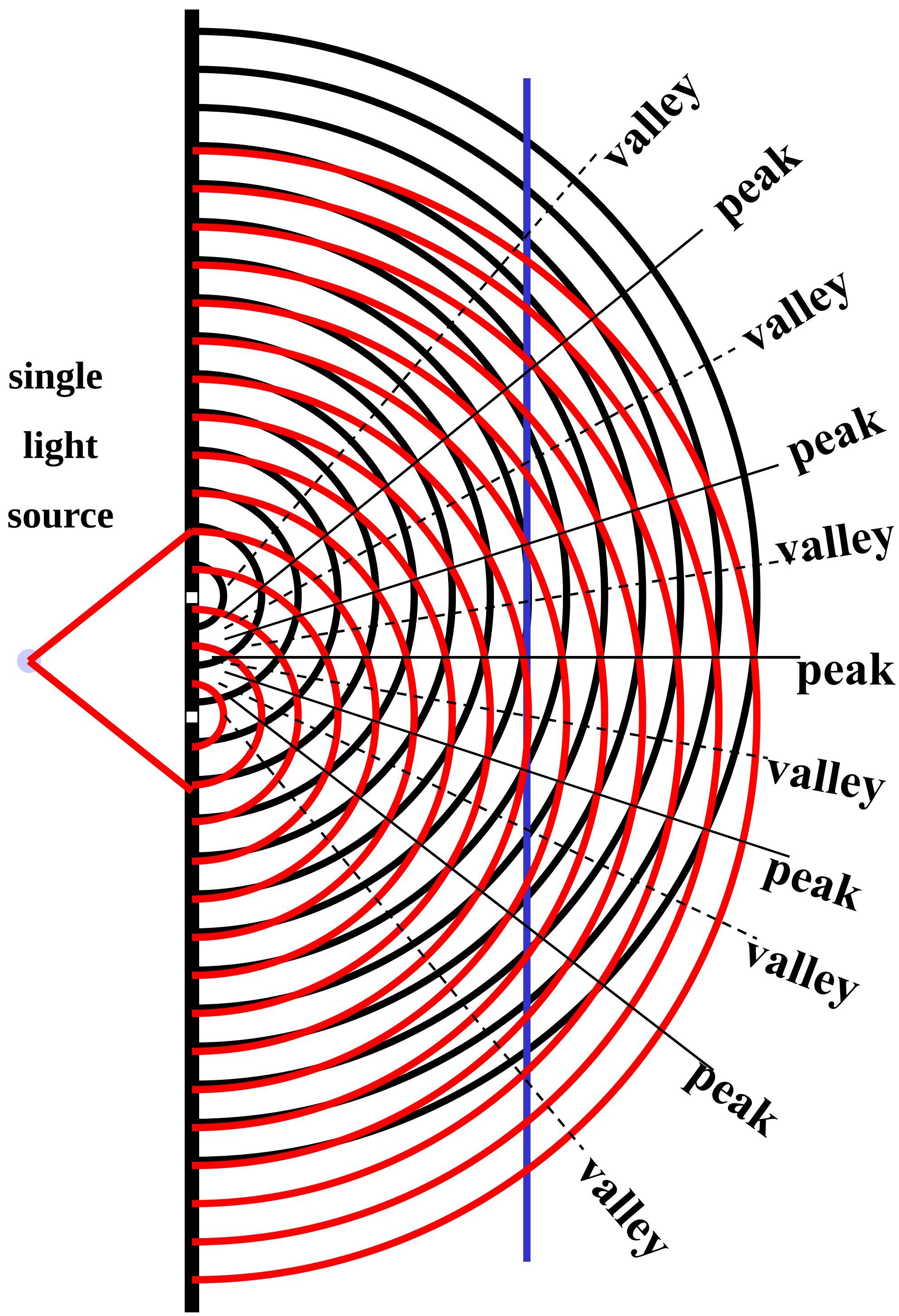
Example:

What is the intensity of two incoherent waves, one with amplitude 5 and another with amplitude 13.

$$I_{incoherent} = E_1^{\ 2} + E_2^{\ 2} = A_1^{\ 2} + A_2^{\ 2} = 5^2 + 13^2 = 194$$

Young's Double Slit Experiment

If you take coherent light from a single light source, you can generate an interference pattern by interfering the light source with itself by passing it through two slits.



Calculation for Young's Double-Slit Experiment

The two light sources originate from the same source, and are mutually coherent. How they interfere with each other depends on the phase differences between the two at any point across the screen. The first step in computing the double slit interference pattern it to compute how the paths length will differ from each other at various parts of the screen. From the diagram we can derive the following:

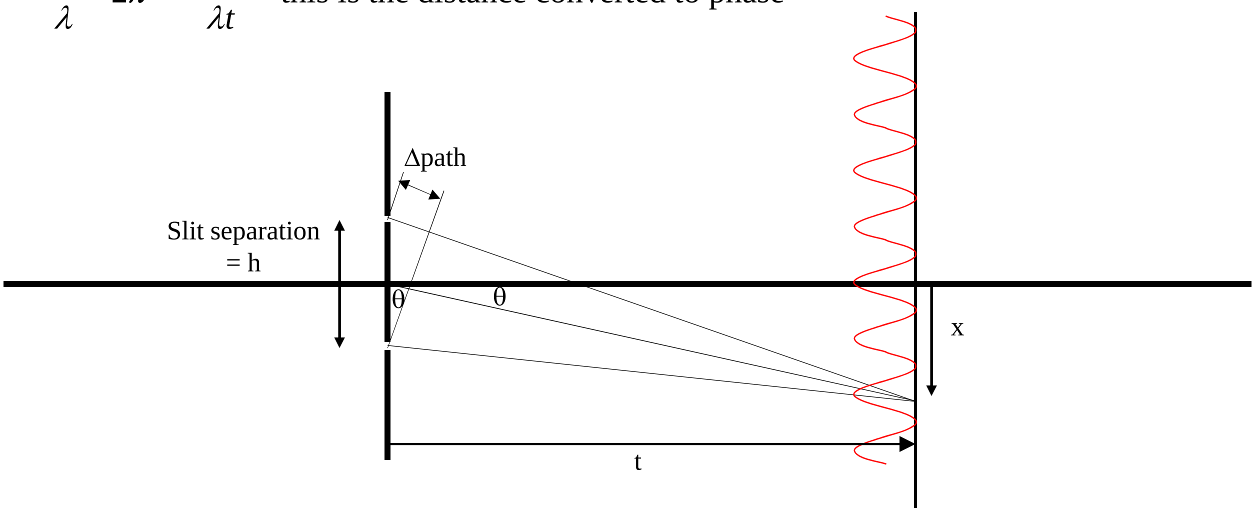
We need to convert the physical difference in distance to a difference in phase, since that is what is important for interference. 1 wavelength difference would be a phase difference of 2π radians. So if we convert the physical difference to number of wavelengths and multiply by 2π then we'll have the phase difference.

$$\sin\theta = \frac{\Delta path}{h} \cong \theta$$

$$\tan\theta = \frac{x}{t} \cong \theta$$

$$\therefore \frac{\Delta path}{h} = \frac{x}{t} \quad \Rightarrow \quad \Delta path = \frac{hx}{t}$$ this is the distance

$$\frac{\Delta path}{\lambda} 2\pi = \frac{2\pi hx}{\lambda t}$$ this is the distance converted to phase



Now recall the formula for the intensity of the superposition of two mutually coherent waves

$$I_{coherent} = \left(E_1 + E_2\right)^2$$

$$= A_1^{\,2} + A_2^{\,2} + 2\cdot A_1 A_2 \cos\left(\alpha_1 - \alpha_2\right)$$

$$= 2\cdot A^2 + 2\cdot A^2 \cos\left(\frac{\Delta path}{\lambda}\cdot 2\pi\right)$$

$$= 2\cdot A^2 + 2\cdot A^2 \cos\left(\frac{2\pi hx}{\lambda t}\right)$$

Substitute in the expression for phase difference that was described above

According to the equation, the maxima will occur when the argument for the cosine is 0, +/-2π, …

or when: $\frac{hx}{\lambda t} = 0, \pm 1, \pm 2 ...$

or when $x = \frac{m\lambda t}{h}, m = 0, \pm 1, \pm 2 ...$

m is a counter. These types of counters are something you will see frequently in the lecture on physical optics

The equation is a good example of one equation with four variables. If you know three of them, then you can figure out the fourth.

Say you have a laser and need to know the wavelength, then you generate the interference pattern with a pair of pinholes at a known distance and you can determine it. It is a very powerful technique. These wavelengths are very short, yet with the right equipment, you can indirectly measure any light wavelength with a normal ruler.

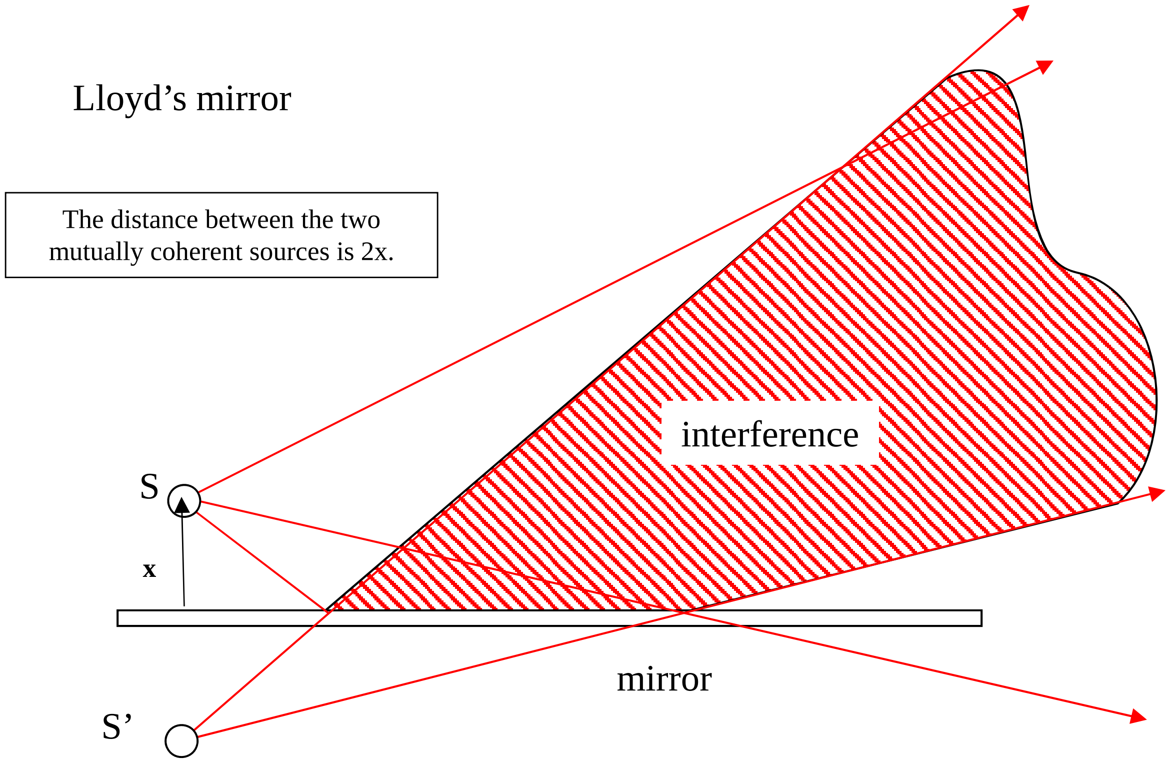
Example: An aperture with a 0.1 mm slit spacing gives a separation between maxima of 1 cm for a wavelength of 500 nm when the screen is held at a distance of 2 meters

$$x = \frac{m\lambda t}{h} = 0 \text{ for m=0, for m} = 1, \ x = \frac{1 \times 500 \times 10^{-9} \times 2}{0.1 \times 10^{-3}} = 0.01 = 1 \text{ cm}$$

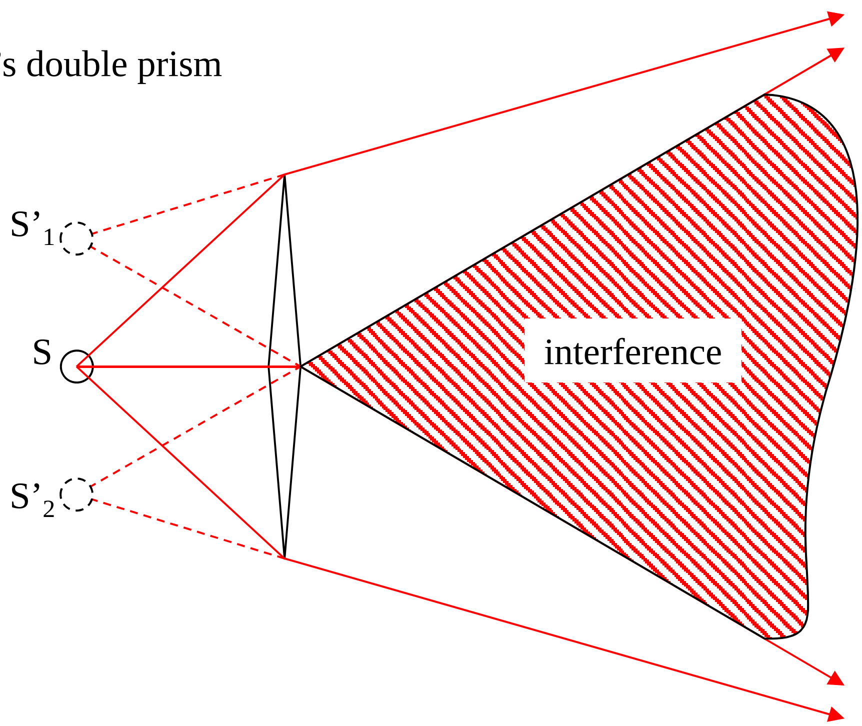
Notice that wavelength is one of the factors governing the distance between maxima. So when the wavelength is longer, the peaks are further apart. This can be seen in a white light double slit interferogram. (I will show a slide of this in the class)

Other Methods to Generate Interference Patterns.

Lloyd's mirror



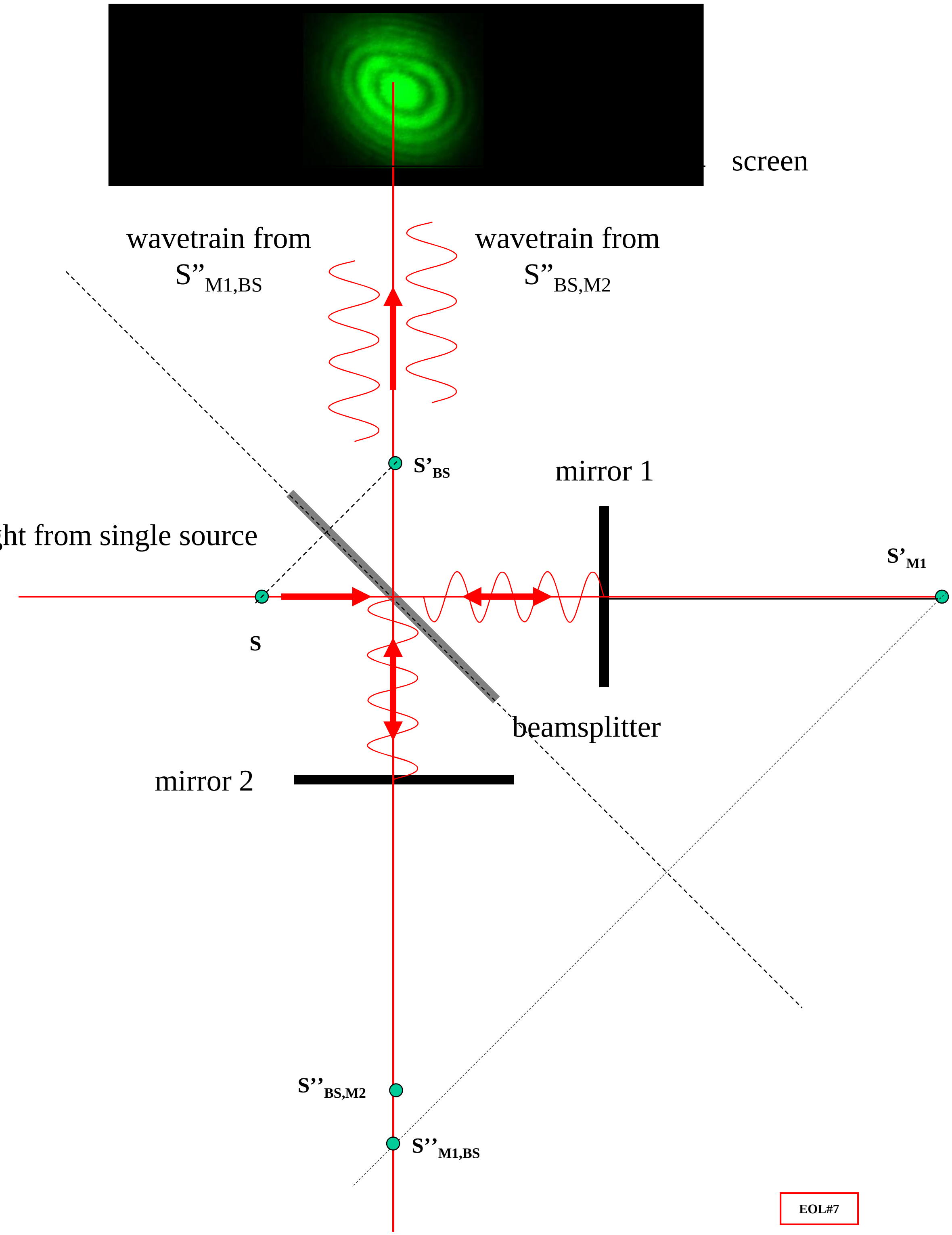
Fresnel's double prism



Uses of Interference:

Interference patterns can be used to make measurements of distances to a very fine scale.

Michelson Interferometer (this will be demonstrated in class):



Demo: E+25+60 Physics demos